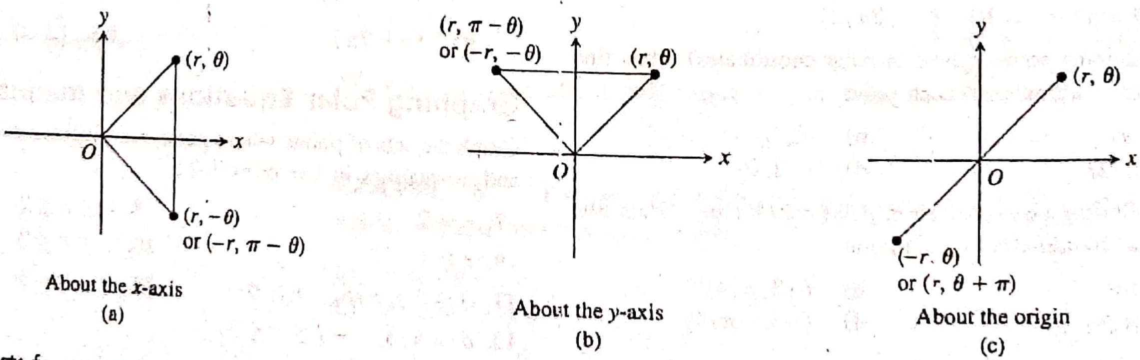
9.7 Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Figure 9.53 illustrates the standard polar coordinate tests for symmetry.



9.53 Three tests for symmetry.

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Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Fig. 9.53a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Fig. 9.53b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Fig. 9.53c).

Slope

The slope of a polar curve $r = f(\theta)$ is given by dy/dx, not by $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

If f is a differentiable function of θ, then so are x and y and, when $dx/d\theta \neq 0$, we can calculate dy/dx from the parametric formula

v.g.p

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad \text{Section 9.5, Eq. (1) with } t = \theta$$

$$= \frac{\dfrac{d}{d\theta}(f(\theta)\cdot\sin\theta)}{\dfrac{d}{d\theta}(f(\theta)\cdot\cos\theta)}$$

$$= \frac{\dfrac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\dfrac{df}{d\theta}\cos\theta - f(\theta)\sin\theta} \qquad \text{Product Rule for Derivatives}$$

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}, \tag{1}$$

provided $dx/d\theta \neq 0$ at (r, θ).

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and Eq. (1) gives

$$\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0.$$

If the graph of $r = f(\theta)$ passes through the origin at the value $\theta = \theta_0$, the slope of the curve there is $\tan\theta_0$. The reason we say "slope at $(0, \theta_0)$" and not just "slope at the origin" is that a polar curve may pass through the origin more than once, with different slopes at different θ-values. This is not the case in our first example, however.

Curve Sketching in Rectangular Coordinates

(7.17) Let an equation of a plane curve in rectangular coordinates be given by $f(x, y) = 0$. To sketch its graph, we examine the following properties of the curve:

1. Symmetry.

(i) If $f(x, -y) = f(x, y)$, the curve is symmetric about the x-axis (Figure 7.17). In this case no odd powers of y occur in the equation of the curve.

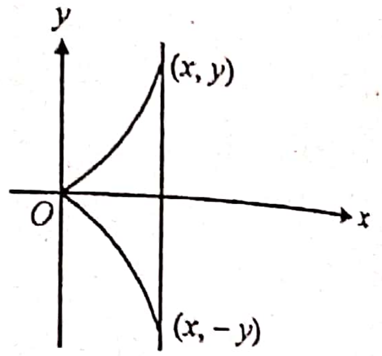

Figure 7.17

(ii) If $f(-x, y) = f(x, y)$, the curve is symmetric about the y-axis (Figure 7.18). This is possible if only even powers of x occur in the equation of the curve.

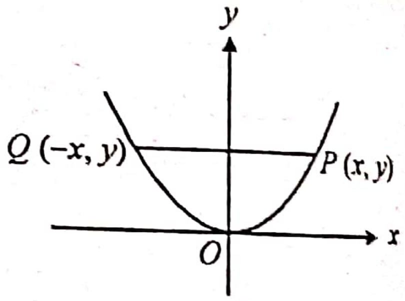

Figure 7.18

(iii) If $f(y, x) = f(x, y)$, the curve is symmetric about the line $y = x$ (Figure 7.19).

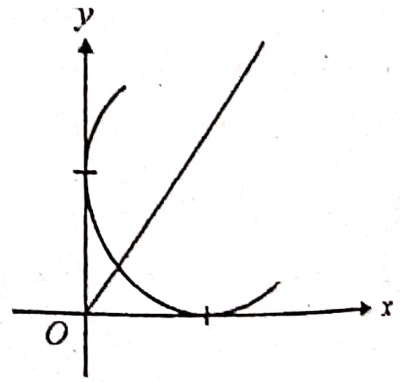

Figure 7.19

(iv) If $f(-x, -y) = f(x, y)$, the curve is symmetric about the origin. The origin in then said to be centre of the curve (Figure 7.20).

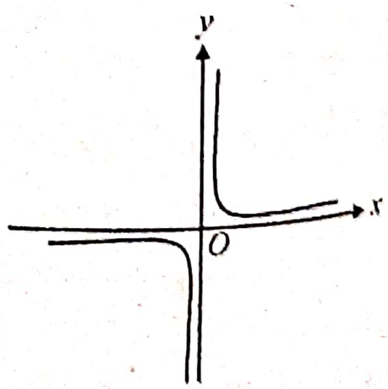

Figure 7.20

II. Check whether the origin lies on the curve. If it does, write down the tangents there at. In case the origin is a multiple point, find out its nature.

III. Find the intercepts of the curve on the coordinate axes. To find the x-intercept, put $y = 0$ in the equation of the curve and solve for x. Similarly, for the y-intercept, put $x = 0$ in the equation and solve for y.

IV. Find out $\frac{dy}{dx}$ and the points where the tangents are parallel to the coordinate axes.

V. Find the multiple points of the curve, if any, and their nature.

VI. Find the asymptotes of the curve.

VII. Determine whether there is any region of the plane such that no part of the curve lies in it.

Example 25. Sketch the curve defined by $3ay^2 = x(x-a)^2$.

Solution.

I. The curve is symmetric about the x-axis.

II. The curve passes through (0, 0) and $x = 0$ is tangent at the origin.

III. The curve has no asymptotes.

IV. $y = \pm \frac{\sqrt{x}(x-a)}{\sqrt{3a}}$. When $x < 0$, y is imaginary. Therefore, the curve does not lie to the left of y-axis.

V. The curve meets the x-axis at $A(a, 0)$. Shifting the origin to $(a, 0)$, we have
$3ay^2 = (x+a)x^2$,
i.e., $x^3 + ax^2 - 3ay^2 = 0$
Therefore, tangents at $(a, 0)$ are

$$x^2 - 3y^2 = 0 \quad \text{i.e.,} \quad y = \pm \frac{1}{\sqrt{3}} x$$

Thus, $(a, 0)$ is a node.
The shape of the curve is as shown in Figure 7.21.

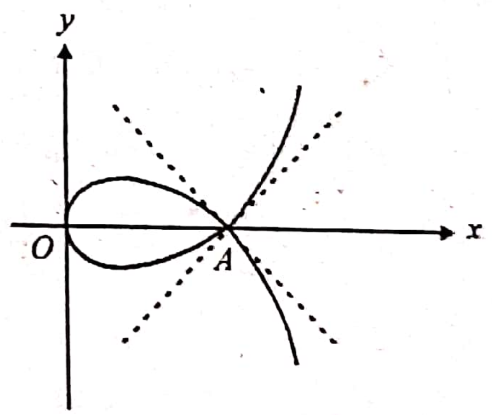

Figure 7.21

Example 26. Sketch the curve

$$y^2(a^2 + x^2) = x^2(a^2 - x^2)$$

Solution. We note the following particulars of the curve:

I. It is symmetric about both the axes.

II. It passes through the origin and $y = \pm x$ are two tangents there at. Thus, the origin is a node.